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EXISTENCE OF THE GENERALIZED GREEN'S FUNCTION

By W. D. A. Westfall

In his second paper on the theory of linear integral equations,* Hilbert has established a relation between these equations and the boundary problems of ordinary, self-adjoined differential equations in the following manner:

If in

$$L(u) \equiv \frac{d}{ds}\left[p(s)\,\frac{du}{ds}\right] + q(s)\,u,$$

$p(s)$ is continuous, together with its first derivative, and $q(s)$ is continuous for $a \leqq s \leqq b$, then

$$\int_a^b vL(u) - uL(v)\,ds = \Big[p(vu' - uv')\Big]_a^b, \tag{1}$$

provided u and v are continuous with their first and second derivatives. This is Green's formula for one dimension. A set of two linear, homogeneous boundary conditions that involve the vanishing of

$$\Big[p(vu' - uv')\Big]_a^b$$

for any two functions u and v, each satisfying these boundary conditions, is a set of Green's boundary conditions.

A continuous solution of

$$L(u) + \lambda u = 0,$$

with a continuous first derivative, and satisfying moreover a set of Green's boundary conditions, is called a characteristic solution. A continuous solution $G(s, t)$ of $L(u) = 0$, satisfying the boundary conditions, but having at $s = t$ the following discontinuity in its first derivative:

$$\operatorname*{Lim}_{\epsilon=0}\left[\frac{dG}{ds}\right]_{s=t+\epsilon} - \operatorname*{Lim}_{\epsilon=0}\left[\frac{dG}{ds}\right]_{s=t-\epsilon} = -\frac{1}{p(t)},$$

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* *Göttinger Nachrichten*, 1904.

is called a Green's function. It follows from (1) that if u_i, u_j are characteristic solutions of

$$L(u_i) + \lambda_i u_i = 0,$$
$$L(u_j) + \lambda_j u_j = 0,$$

$\lambda_i \neq \lambda_j$, then

$$\int_a^b u_i(s)\ u_j(s)\ ds = 0. \tag{2}$$

With the Green's function $G(s, t)$ they satisfy the relation

$$u_i(s) = \lambda_i \int_a^b G(s, t)\ u_i(t) dt. \tag{3}$$

Moreover any function satisfying (3) with a Green's function $G(s, t)$ is such a characteristic solution of $L(u) + \lambda_i u = 0$. If the Green's function exists, the relation between integral equations and the boundary problems of differential equations is thus established.

If $p(s)$ is positive in $a \leqq s \leqq b$, it is known that either a Green's function or a characteristic solution of $L(u) = 0$ exists, but not both.* In case a characteristic solution exists, in which case there is no ordinary Green's function, Hilbert has defined a generalized Green's function $H(s, t)$, which replaces G in (3). If ω_1, ω_2 constitute a complete set of linearly independent solutions of $L(u) = 0$, normalized and orthogonalized, that is, such that

$$\int_a^b \omega_1 \omega_2 ds = 0, \quad \int_a^b \omega_1^2 ds = 1, \quad \int_a^b \omega_2^2 ds = 1,$$

then the generalized Green's function is the Green's solution $H(s, t)$ of

$$L(u) = \omega_1(s)\ \omega_1(t) + \omega_2(s)\ \omega_2\ (t), \dagger \tag{4}$$

for which

$$\int_a^b H(s, t)\ \omega_n(t)\ dt = 0, \qquad n = 1, 2. \tag{5}$$

There follow in the article referred to certain examples of this generalized Green's function, but no attempt is made to prove its existence in general. It is the purpose of this note to supply this proof.

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* The writer's dissertation, Göttingen, 1905, p. 20.

† Hilbert considers only the case where only a single characteristic solution exists for $\lambda = 0$.

Let the two linearly independent solutions of $L(u) + \lambda u = 0$ be $z_1(\lambda, s)$ and $z_2(\lambda, s)$. These are analytic in λ.* If a characteristic solution exists for all values of λ it is given by

$$c_1 z_1(\lambda, s) + c_2 z_2(\lambda, s) = u(\lambda, s),$$

where c_1 and c_2 are so determined that the boundary conditions are satisfied. Hence such a solution is analytic in λ. From (2)

$$\int_a^b u(\lambda, s)\ u(\lambda + h, s)\ ds = 0,$$

or, by the theorem of the mean,

$$\int_a^b [u(\lambda, s)]^2\, ds + h\int_a^b u(\lambda, s)\, \frac{\partial u(\lambda + \theta h, s)}{\partial \lambda}\, ds = 0, \qquad 0 < \theta < 1,$$

which is impossible for h sufficiently small, unless $u(\lambda, s) = 0$.

There exists then for some value of λ, as $\lambda = \rho$, no characteristic solution, but a Green's function $G(\rho; s, t)$, since $L(u) + \rho u = 0$ is an equation of the same type as $L(u) = 0$, where $q + \rho$ replaces q. Let

$$E(\rho;\ s, t) = G(\rho;\ s, t) + \frac{1}{\rho}\left[\omega_1(s)\,\omega_1(t) + \omega_2(s)\,\omega_2(t)\right].$$

E is a Green's function of

$$L(u) + \rho u = \omega_1(s)\,\omega_1(t) + \omega_2(s)\,\omega_2(t),$$

such that

$$(6) \qquad \int_a^b E(\rho;\ s, t)\ \omega_n(s)\ ds = 0, \qquad n = 1, 2.$$

From the theory of linear integral equations we know† that there is a continuous solution $K(s, t)$ of

$$(7) \qquad E(\rho;\ s, t) = K(s, t) + \rho\int_a^b E(\rho;\ s, r)\ K(r, t)\ dr,$$

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* Picard: *Traité d'Analyse*, vol. 2, p. 92.

† I. Fredholm: *Acta Mathematica*, vol. 27 (1903).

or a not-identically vanishing solution of

$$\phi(s) = -\rho \int_a^b E(\rho\,;\, s,\, t)\, \phi(t)\, dt.$$

From (6) a solution of this latter equation is orthogonal to ω_1 and ω_2, that is

$$\int_a^b \omega_n(s)\, \phi(s)\, ds = 0, \qquad n = 1,\, 2. \tag{8}$$

It satisfies, moreover, all linear homogeneous boundary conditions satisfied by E in s, and is a solution of $L(u) = 0$, as may be seen by direct substitution. It must then be a characteristic solution of $L(u) = 0$. But equation (8) shows that such a function ϕ cannot exist, since ω_1 and ω_2 constitute a complete set of linearly independent characteristic solutions. There exists then a continuous solution $K(s, t)$ of (7). It is seen immediately that this is the generalized Green's function sought.

It has been assumed in the above proof that two linearly independent characteristic solutions of $L(u) = 0$ exist. Neglecting the case where only one exists has affected the proof only in simplifying the notation. The following theorem has been shown:

If $p(s)$ is continuous together with its first derivative, and positive, and if $q(s)$ is continuous in a given closed interval, there exists for each set of Green's boundary conditions for this interval either a Green's solution or a generalized Green's solution of $L(u) = 0$.

This proof, with a mere change in notation, holds for self-adjoined linear differential equations of the $2n$th order.

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